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Nonlinear Analysis of Thin Homogeneous Orthotropic Elastic Plates under Large Deflection and Thermal Loading

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Abstract—The present investigation deals with the nonlinear deflection of homogeneous elastic plates of uniform thickness made of an orthotropic material, under uniform heating. The governing partial differential equations are deduced by the method of constant deflection contour lines proposed by Majumdar, and taking into account the modified approach of Banerjee and Sinharoy. The final solution has been obtained by the method of Galerkin. The numerical results have been shown graphically and compared with other available results, both for elliptic and circular plates with immovable as well as movable edge conditions.

1. INTRODUCTION

On going through the literature, it is found that very few works have been done in nonlinear thermoelasticity, so far as large deflection of anisotropic elastic bodies are concerned. This is due to, in part, the mathematical complexity of such problems, but also to the fact that engineering structures have generally been fabricated from materials which are essentially isotropic. However, owing to the increased uses of macroscopically anisotropic construction materials, like fibre reinforced composites, in situations involving severe thermal environments, further study in the area is required. The large deflection of a thin elastic plate is governed by two coupled, nonlinear partial differential equations, which were initially studied by Von Karman [1] and further reformulated by Vinokurov [2] to include the effect of temperature. The general solution of these equations is unknown either in elastic or thermoelastic problems. As a result, several approximate methods were suggested by different investigators [3–6]. One of these approximate methods, proposed by Berger [3], in spite of its inherent deficiencies, as rightly pointed out by Nowinski and Ohnabi [7], yielded good results in some aspects. The rectification of Berger's method was proposed by Banerjee and Sinharoy [6] in the case of movable edges. Indeed, the study of large deflections of elastic and thermoelastic plates has become increasingly important, particularly in the analysis of structures used in aircrafts, so several attempts have been made to obtain a simple theory.

The purpose of the present investigation is to develop a simple, and yet sufficiently accurate, method for the analysis of large deflections of a thin homogeneous, orthotropic plate in thermoelasticity, following the method of Majumdar [5]. The method of constant deflection contour

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lines of Majumdar [5] is used, in conjunction with the modification of Banerjee and Sinharoy [6] to obtain a new set of nonlinear partial differential equations for the analysis of large deflections in heated anisotropic elastic plates of arbitrary shape, overcoming the shortcomings of Berger's method. Although the discussion is confined to the case of an elliptic and a circular plate, considering the elliptic plate as a model, application of the method to other boundary shapes is straightforward.

The present modified approach seems more advantageous in the analysis of large deflections of orthotropic elastic plates than those described elsewhere. Here, the results for the nonlinear characteristics of different plates with movable and immovable edge conditions can be obtained from a single differential equation, and the method of solution provides accurate results with much less computational effort.

2. FORMULATION OF THE PROBLEM

Let us consider the large deflection of a thin homogeneous heated anisotropic elastic plate of uniform thickness h . The middle plane of the plate is taken to be the xy -plane, coinciding with the horizontal plane, and the z -axis is directed downwards with reference to a system of orthogonal coordinates xyz . Let the transverse displacement of a point to the middle plane be denoted by w , which is a function of the spatial coordinates (x, y) . The interactions of the deflection surface $z = w(x, y)$ with the parallel $z = \text{constant}$ yield contours which, after projection onto the xy -plane, are a set of level curves $u(x, y) = \text{constant}$ called *lines of equal deflection*. If the boundary C of the plate is subjected to any combination of clamping and simple support, so that it does not move in the direction perpendicular to the plane of the plate, then clearly the boundary will belong to the family of *lines of equal deflection*, and without loss of generality one may consider $u = 0$ on the boundary of the plate.

Let us denote the family of curves $u = \text{constant}$ by C_u , $0 \leq u \leq u^*$, so that $C_0 = C$ is the boundary of the plate; C_{u^*} coincides with the point at which the maximum $u = u^*$ is attained.

Let us consider the equilibrium of an element Ω_u of the plate bounded by any closed contour C_u ; summing up the forces in the vertical direction yields the following statical equation:

$$\oint_{C_u} \left(Q_n - \frac{\partial M_{nt}}{\partial s} \right) ds + \iint_{\Omega_u} \left(q + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) d\Omega = 0, \quad (2.1)$$

where the line integral represents the upward vertical contribution of the resultant tractions exerted upon the portion by the remainder, N_x , N_y , N_{xy} represent the membrane forces acting on a small element lying within the contour C_u , Q_n is the shearing force, and M_{nt} the twisting moment along the contour.

Many authors applied Berger's method for the investigation of large deflection in plates and shells. The essential feature of Berger's method is in its decoupling of the two nonlinear Von Karman's equations, so that one of them assumes a quasilinear form and can readily be integrated (in spite of the lack of the mechanical interpretation of doing so). Goldberg [4] proposed another modified method for the analysis of large deflections of plates. Following the methods of Berger and Goldberg, Basuli [8], Mukhopadhyay and Bera [9], respectively, analysed the large deflection of a heated plate. As Berger's as well as Goldberg's methods are ineffective in the case of a movable edge boundary condition, Banerjee and Sinharoy [6] proposed another approach modifying the method of Berger and Goldberg and obtained excellent results for the movable edge boundary condition for isotropic materials. After going through the literature, it has been observed that not much work has been done yet in the field of anisotropic thermoelasticity in the line of the present analysis.

Using the Duhamel-Neumann law for a thin plate made of an orthotropic material subjected to a plane state of stress perpendicular to the z -axis, the stress-strain relations in orthotropic

thermoelasticity can be written [10] as

$$\begin{aligned}\tau_{11} &= \frac{E_{11}(e_{11} + \nu_{21} e_{22} - (\alpha_{11} + \nu_{21} \alpha_{22})\theta)}{(1 - \nu_{12} \nu_{21})}, \\ \tau_{22} &= \frac{E_{22}(e_{22} + \nu_{12} e_{11} - (\alpha_{22} + \nu_{12} \alpha_{11})\theta)}{(1 - \nu_{12} \nu_{21})}, \\ \tau_{12} &= 2G_{12} e_{12},\end{aligned}\quad (2.2)$$

where the coefficients α_{ij} characterised a linear thermal expansion of an anisotropic type, E_{ii} are called the Young's moduli associated with the directions x , y , and G_{12} is known as the shear modulus associated with the plane xy , ν_{ij} are the Poisson's coefficients describing the contraction in the i -direction produced by the tension in the j -direction, τ_{ij} and e_{ij} , $i, j = 1, 2$ are stresses and strains, respectively, and θ is the temperature.

Introducing the modification of Banerjee and Sinharoy [6] to the simplified approach of Majumdar [5] in equation (2.1) with the help of (2.2), and substituting the well-known expressions for Q_n and M_{nt} for orthotropic thermoelasticity, one obtains the statical equation in the following form:

$$\begin{aligned}\frac{d^3 w}{du^3} \oint_{C_u} R_0 ds + \frac{d^2 w}{du^2} \oint_{C_u} F_0 ds + \frac{dw}{du} \oint_{C_u} G_0 ds - \iint_{\Omega_u} \left[q + \frac{12 D_1 A}{h^2} (w_{xx} + \nu_{21} w_{yy}) \right. \\ \left. - D_1 (\alpha_{11} + \nu_{21} \alpha_{22}) \frac{\partial^2 \theta_1}{\partial x^2} - D_2 (\alpha_{22} + \nu_{21} \alpha_{11}) \frac{\partial^2 \theta_1}{\partial y^2} \right. \\ \left. + \frac{6\lambda D_1}{h^2} ((w_x^2 + w_y^2) \nabla^2 w + 2 (w_x^2 w_{xx} + w_y^2 w_{yy} + 2w_x w_y w_{xy})) \right. \\ \left. - \frac{12 D_1}{h^2} \alpha_{22} \left(w_x \frac{\partial \theta_0}{\partial x} + w_y \frac{\partial \theta_0}{\partial y} + \theta_0 \nabla^2 w \right) \sqrt{\lambda (k^2 - \nu_{21}^2)} \right] dx dy = 0,\end{aligned}\quad (2.3)$$

with

$$R_0 = -\frac{D_1}{\sqrt{t}} (u_x^4 + 2l^2 u_x^2 u_y^2 + k^2 u_y^4), \quad (2.4)$$

$$F_0 = -\frac{D_1}{\sqrt{t}} (3 (u_{xx} u_x^2 + k^2 u_{yy} u_y^2) + l^2 (u_x^2 u_{yy} + u_y^2 u_{xx}) + 4l^2 u_x u_y u_{xy}), \quad (2.5)$$

$$\begin{aligned}G_0 = -\frac{D_1}{t^{3/2}} \left[u_x^3 u_{xxx} + k^2 u_y^3 u_{yyy} + (1 + p^2) u_x u_y^2 u_{xxx} + (k^2 + p^2) u_x^2 u_y u_{yyy} \right. \\ \left. + (l^2 + p^2) (u_x^3 u_{xyy} + u_y^3 u_{xxy}) + (l^2 - 2p^2) (u_x u_y^2 u_{xyy} + u_y u_x^2 u_{xxy}) \right. \\ \left. - 2p^2 u_{xy} (u_x u_y (u_{xx} + u_{yy}) - (u_x^2 + u_y^2) u_{xy}) + p^2 (u_{xx} - u_{yy}) (u_{xx} u_y^2 - u_{yy} u_x^2) \right] \\ + \frac{2D_1 p^2}{t^{5/2}} [u_{xy} (u_x^2 - u_y^2) - u_x u_y (u_{xx} - u_{yy})]^2,\end{aligned}\quad (2.6)$$

where

$$\begin{aligned}t &= u_x^2 + u_y^2, & p^2 &= \frac{2D_k}{D_1}, & l^2 &= \frac{(\nu_{21} D_1 + 2D_k)}{D_1}, & D_k &= \frac{G_{12} h^3}{12}, \\ D_1 &= \frac{E_{11} h^3}{12 (1 - \nu_{12} \nu_{21})}, & D_2 &= \frac{E_{22} h^3}{12 (1 - \nu_{12} \nu_{21})}, \\ k^2 &= \frac{\nu_{21}}{\nu_{12}} = \frac{E_{22}}{E_{11}} = \frac{D_2}{D_1}, & \theta &= \theta_0(x, y) + z \theta_1(x, y),\end{aligned}$$

and the normalised constant of integration A is determined from

$$\begin{aligned}A + (\alpha_{11} + \nu_{21} \alpha_{22})\theta_0 &= e = e_{11} + \nu_{21} e_{22} \\ &= \frac{\partial u_1}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \nu_{21} \frac{\partial v}{\partial y} + \frac{\nu_{21}}{2} \left(\frac{\partial w}{\partial y} \right)^2,\end{aligned}\quad (2.7)$$

by integrating over the whole area of the plate, with u_1 and v being the components of displacements in x and y -directions, respectively.

In particular, equations (2.3) to (2.7) coincide with those of Banerjee and Sinharoy [6], if we substitute $E_{11} = E_{22} = E$, $\nu_{12} = \nu_{21} = \nu$, $p^2 = 1 - \mu$, $D_1 = D$, $k^2 = l^2 = 1$. Further, these equations can be identified with those of Majumdar and Jones [11] for immovable isotropic case, if we substitute $\nu_{21} = 1$ in (2.7) and in the coefficient of A in (2.3), $\lambda = 0$ in (2.3) and $\theta_1 = \theta_0 = 0$, $E_{11} = E_{22} = E$, $\nu_{12} = \nu_{21} = \nu$, $p^2 = 1 - \mu$, $D_1 = D$, $k^2 = l^2 = 1$.

In this connection, it may be mentioned that there are some misprints in [11] in the expressions corresponding to F_0 and G_0 , and some mistakes in [6] in the equation corresponding to (2.3).

Large Deflection Analysis for an Elliptic Plate

Let us consider the problem of an elliptic plate whose semimajor and semiminor axes are designated by a and b , respectively, and the edges are clamped. As a first approximation, let us assume the lines of equal deflection to be a family of similar and similarly situated ellipses and, hence, we may take $u(x, y) = 1 - x^2/a^2 - y^2/b^2$.

Thus, calculating the values of R_0 , F_0 , and G_0 , substituting these in equation (2.3), integrating and taking $f^2 = 1 - u$, we obtain, from (2.3), the differential equation

$$f \frac{d}{df} \left(\frac{1}{f} \frac{d}{df} \left(f \frac{d}{df} \right) - 4(M^2 - N^2) \right) w - \frac{6\lambda B}{h^2} f \left(\frac{dw}{df} \right)^3 - 8q_2 f^2 = 0, \quad (2.8)$$

where

$$\begin{aligned} M^2 &= \frac{12 a^2 b^2 A (\nu_{21} a^2 + b^2)}{h^2 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)}, \\ N^2 &= \frac{12 \alpha_{22} a^2 b^2 (a^2 + b^2) \theta_0}{h^2 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)} \sqrt{\lambda (k^2 - \nu_{21}^2)}, \\ B &= \frac{(3b^4 + 3a^4 + 2a^2 b^2)}{(3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)}, \quad q_2 = \frac{a^4 b^4 q'}{2D_1 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)}, \\ q' &= q - D_1 (\alpha_{11} + \nu_{21} \alpha_{22}) \frac{\partial^2 \theta_1}{\partial x^2} - D_2 (\alpha_{22} + \nu_{21} \alpha_{11}) \frac{\partial^2 \theta_1}{\partial y^2}; \end{aligned} \quad (2.9)$$

θ_0 and $D_1 (\alpha_{11} + \nu_{21} \alpha_{22}) \frac{\partial^2 \theta_1}{\partial x^2} + D_2 (\alpha_{22} + \nu_{21} \alpha_{11}) \frac{\partial^2 \theta_1}{\partial y^2}$ are assumed to be constant.

3. SOLUTION OF THE PROBLEM

We now solve equation (2.8) using Galerkin's technique. We assume the deflection function w satisfying the clamped edge boundary condition as

$$w = w_0 (1 - f^2)^2. \quad (3.1)$$

Substitution for w in (2.8) yields an error function ε that is orthogonal over the plate, so

$$\int_0^1 \varepsilon f w df = 0, \quad (3.2)$$

which yields

$$\left(\frac{1}{6} + \frac{1}{20} (M^2 - N^2) \right) \frac{w_0}{h} + \frac{\lambda B}{7} \left(\frac{w_0}{h} \right)^3 - \frac{q_2}{24h} = 0. \quad (3.3)$$

Equation (3.3) determines the central deflection w_0 . Having thus determined the deflection w , the constant A can be determined from (2.7) by substituting (3.1) into this equation and integrating over the area of the plate, which yields

$$A + (\alpha_{11} + \nu_{21} \alpha_{22}) \theta_0 = \frac{w_0^2 (b^2 + \nu_{21} a^2)}{3a^2 b^2}. \quad (3.4)$$

Substituting for A in (3.3), we obtain

$$\left(\frac{w_0}{h}\right) A_1 + \left(\frac{w_0}{h}\right)^3 A_2 = A_3, \quad (3.5)$$

where

$$\begin{aligned} A_1 &= 1 - 18 a^2 b^2 \theta_0 \left[(\nu_{21} a^2 + b^2) (\alpha_{11} + \nu_{21} \alpha_{22}) + \frac{\alpha_{22} \sqrt{\lambda} \sqrt{k^2 - \nu_{21}^2} (a^2 + b^2)}{5 h^2 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)} \right], \\ A_2 &= \frac{6\lambda (3b^4 + 3a^4 + 2a^2 b^2)}{7 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)} + \frac{6 (\nu_{21} a^2 + b^2)^2}{5 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)}, \quad \text{and} \\ A_3 &= \frac{3a^4 b^4 (1 - \nu_{12} \nu_{21}) q'}{2E_{11} h^4 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)}. \end{aligned} \quad (3.6)$$

The deflection equation for the *movable edge* condition is

$$\left(\frac{w_0}{h}\right) A_1 + \left(\frac{w_0}{h}\right)^3 A_2 = A_3, \quad (3.7)$$

where

$$\begin{aligned} A_1 &= 1 - 18 a^2 b^2 \theta_0 \left[\frac{\alpha_{22} \sqrt{\lambda} \sqrt{k^2 - \nu_{21}^2} (a^2 + b^2)}{5 h^2 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)} \right], \\ A_2 &= \frac{6\lambda (3b^4 + 3a^4 + 2a^2 b^2)}{7 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)} \quad \text{and} \\ A_3 &= \frac{3a^4 b^4 (1 - \nu_{12} \nu_{21}) q'}{2E_{11} h^4 (3b^4 + 3k^2 a^4 + 2l^2 a^2 b^2)}. \end{aligned} \quad (3.8)$$

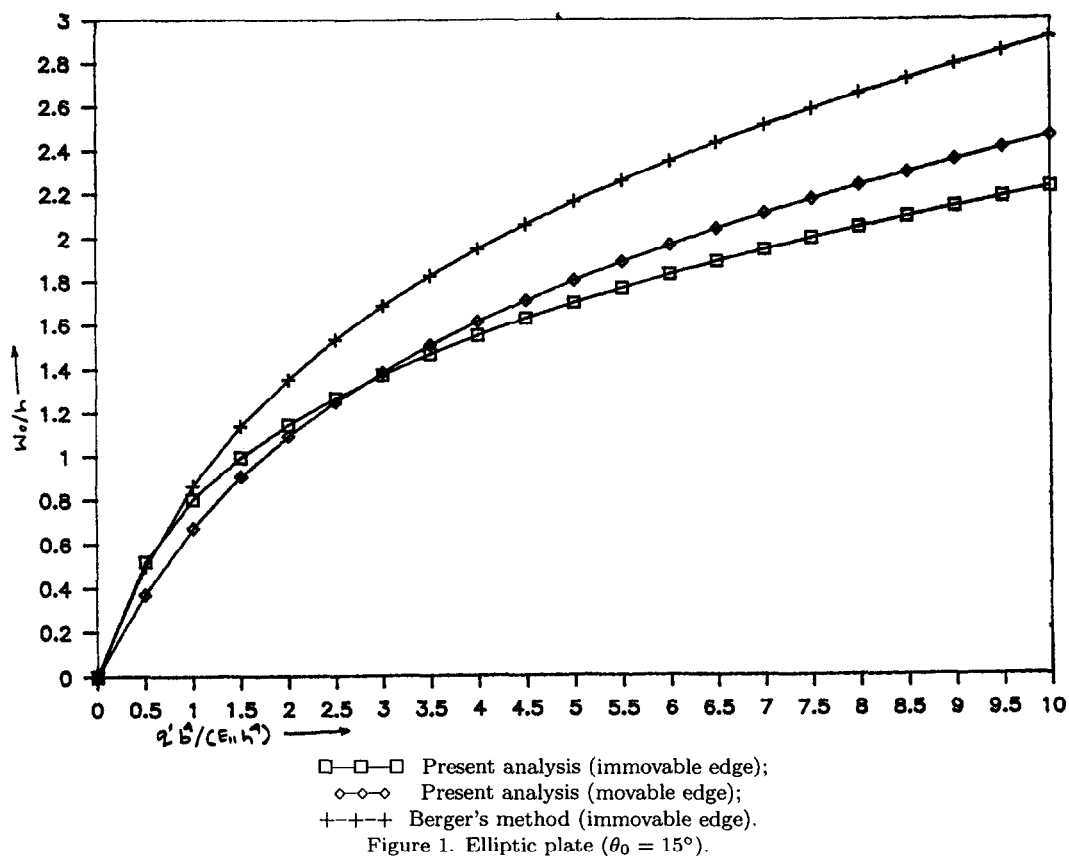
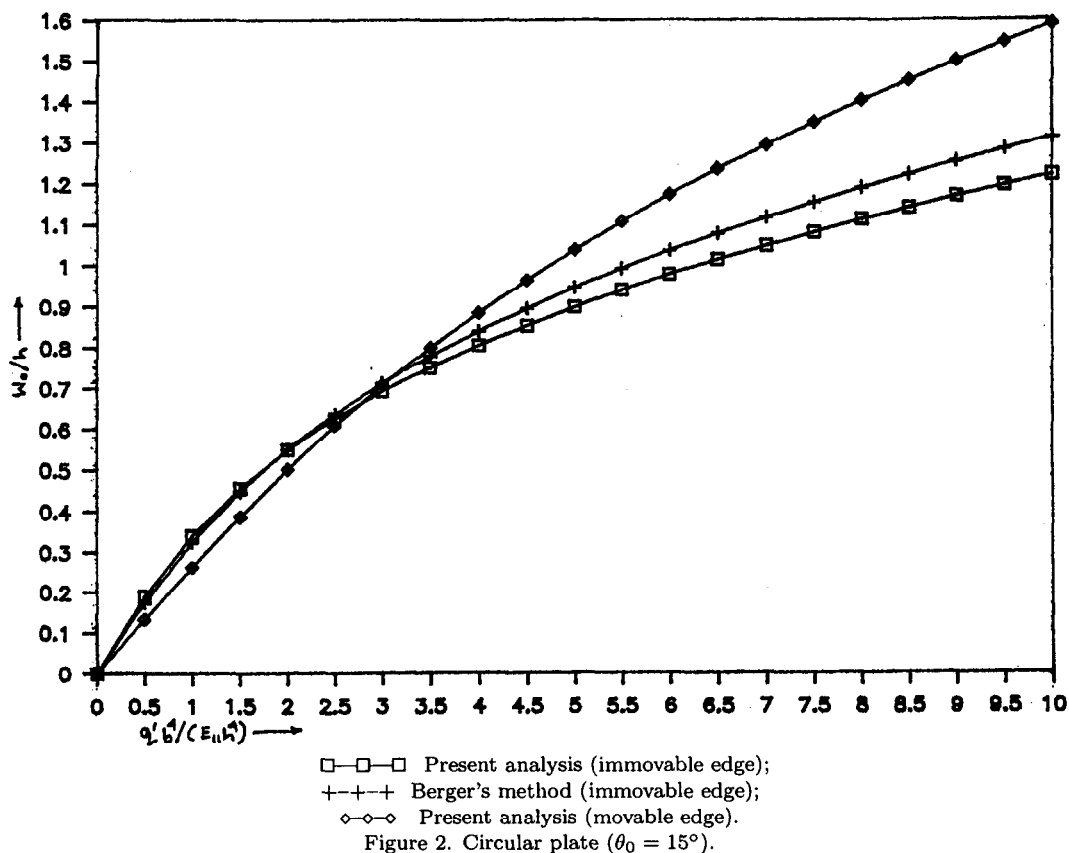
For the *immovable edge* condition, Berger's technique yields the deflection equation (3.5), where A_1 , A_2 , and A_3 are to be determined from (3.6) by substituting $\lambda = 0$. But no relation can be obtained for the *movable edge* condition from Berger's method.

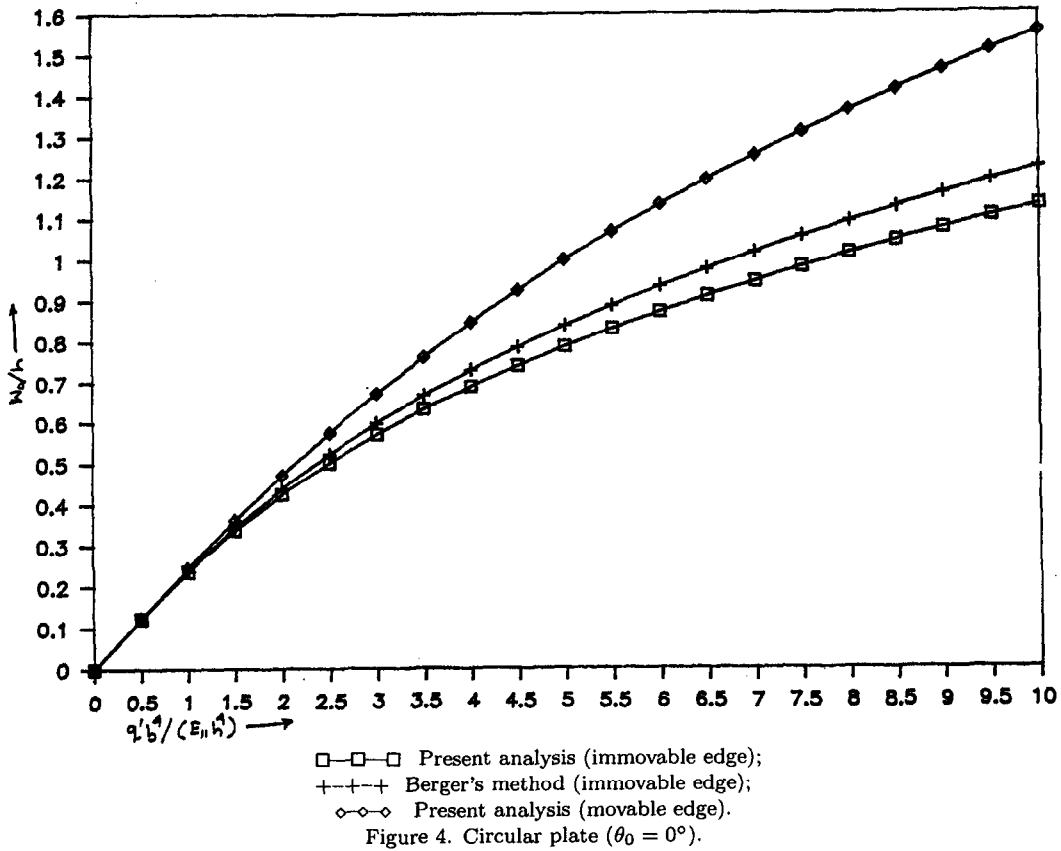
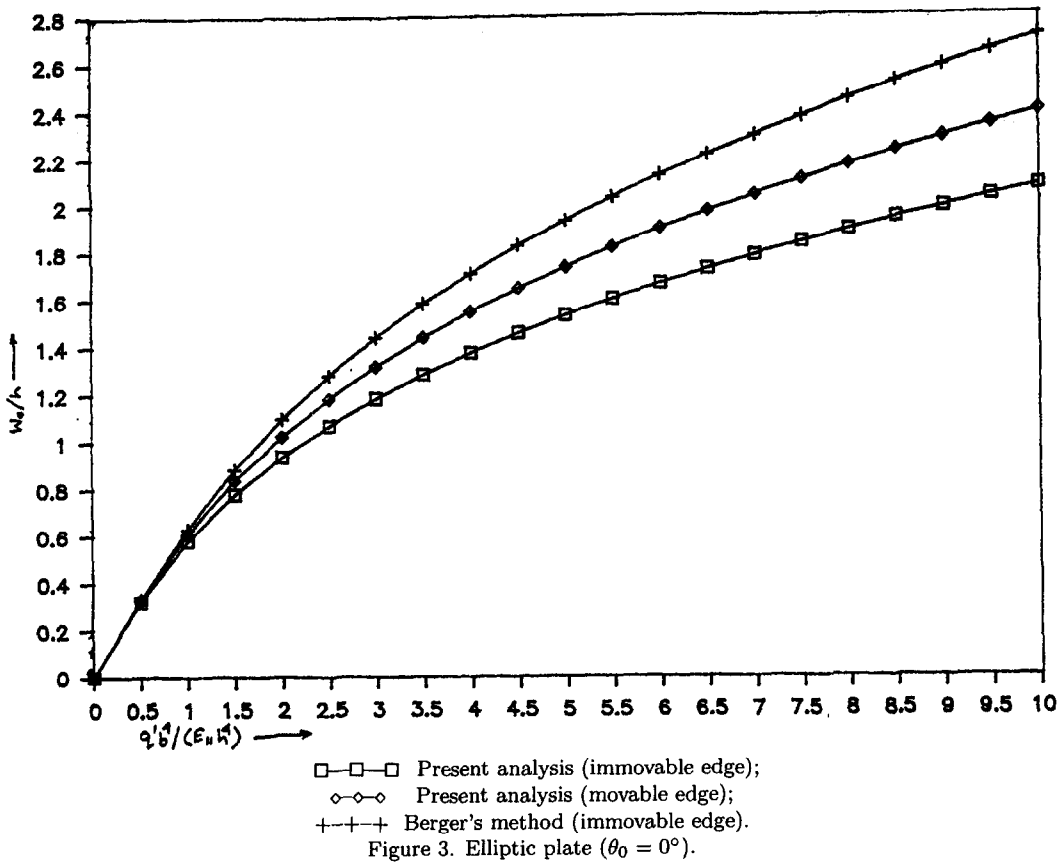
4. NUMERICAL RESULTS

Numerical results are obtained and the graphs of w_0/h are drawn against the thermal loading $q' b^4 / (E_{11} h^4)$ for movable and immovable edges both for elliptic and circular plates, taking the values of $\nu_{21} = 0.3$, $\lambda = 2\nu_{21}^2$, $\alpha_{22} = 4.8 \times 10^{-4}$, $k^2 = 0.5$, $l^2 = 0.2228$, $\alpha_{11}/\alpha_{22} = 1.5$. In Figures 1 and 2, the graphs of w_0/h are drawn against $q' b^4 / (E_{11} h^4)$ taking $\theta_0 = 15^\circ$, whereas in Figures 3 and 4, the corresponding graphs are drawn for $\theta_0 = 0^\circ$. The graphs corresponding to Berger's method are also drawn for comparison, for the immovable edge condition only.

5. OBSERVATIONS AND CONCLUSIONS

The advantage of the present problem is that the results both for immovable as well as movable edge conditions can be obtained from the same governing equation. Furthermore, the problem of an isotropic material can be derived from this problem very easily. It is also observed that the results of the present study are sufficiently accurate for both immovable and movable edge conditions. For elliptic plates, the maximum numerical difference in the different values of the dimensionless thermal load given by Berger's approach and the present study is not very high, although the difference in percentage is pretty high. In the case of circular plates, it is seen from the graph that the difference of the results in the present study and Berger's method is also found to be significant. This is due to the fact that Berger's approach is purely an approximate one based on neglecting the so-called second invariant e_2 in the expression for the potential energy. Thus, the results of the present analysis are more accurate. So, the results for a class of problems related to elastic plates can be obtained from a single differential equation, for different choices of u . Thus the method described here seems more advantageous than those previously reported in the literature. Finally, it may also be noted that the differential equation to be solved here is only of third order.

Figure 1. Elliptic plate ($\theta_0 = 15^\circ$).Figure 2. Circular plate ($\theta_0 = 15^\circ$).



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